Renormalized non-modal theory of the kinetic drift instability of plasma shear flows

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Abstract

The linear and renormalized nonlinear kinetic theory of drift instability of plasma shear flow across the magnetic field, which has the Kelvin's method of shearing modes or so-called non-modal approach as its foundation, is developed. The developed theory proves that the time-dependent effect of the finite ion Larmor radius is the key effect, which is responsible for the suppression of drift turbulence in an inhomogeneous electric field. This effect leads to the non-modal decrease of the frequency and growth rate of the unstable drift perturbations with time. We find that turbulent scattering of the ion gyrophase is the dominant effect, which determines extremely rapid suppression of drift turbulence in shear flow.

52.35.Ra 52.35.Kt

I.INTRODUCTION

After seminal investigations of the regimes of the enhanced plasma confinement[1], the understanding the role of flow shears in turbulent transport becomes one of the major issues of tokamak plasmas physics[2]. The stabilizing effect of the flow shears on various drift waves is recognized as one of the essential elements in the formation of core and edge transport barriers. A decisive step in understanding the role of flow shears in the reducing of the turbulent transport and formation of transport barriers has been made with the development of the gyrokinetic theory for describing turbulence in plasma with large shear flows [3]-[7] and the development of numerical codes[8]-[10] able to solve the set of gyrokinetic equations. In spite of the great progress in numerical investigations of tokamak turbulence, there still remain some key issues in analytical investigations of the turbulence in plasma shear flows, which deserve further clarification. One of the most challenging problem in the theory of plasma shear flows turbulence is the development of analytical methods of the investigations of the long-time evolution of the turbulence in shear flow governed by Vlasov-Maxwell system. Applications of the gyrokinetic approach to analytical investigation of the instabilities in plasma shear flows has been made in series of papers [11]-[13], in which the electrostatic ion-temperature-gradient mode was considered. In all these papers, the spectral transform in time was applied and perturbed

electrostatic potential Φ was considered in canonical modal form, $\Phi \sim \exp(-i\omega t)$. That modal form, however, unsatisfactory represents the long-time response of plasma shear flows across the magnetic field due to space-dependent Doppler shift and stretching of waves pattern by shear flows[14]. In fact, that approach gives results which are valid only for times limited by the condition $t \ll (V_0')^{-1}$, where V_0' is flow velocity shear.

One of the most effective approaches to the analysis of the temporal evolution of plasma turbulence in shear flows is method of shearing modes or so-called non-modal approach. The essence of this approach, which originally was developed by Lord Kelvin[15] for fluid flows with a homogeneous velocity shear, consists in transforming the independent spatial variables from the laboratory frame to a frame convected with shear flow and studying the temporal evolution of the spatial Fourier modes of perturbation without any spectral expansions in time ([14], [16] and references therein). The transformation to the coordinates convected with shear flow eliminates the explicit spatial dependence related to shear flow from the convective derivative in governing fluid equations. This transformation not only simplifies governing equations, but is also principally indispensable. The temporal evolution of a separate spatial Fourier harmonic with a definite wave number can only be analyzed with convective coordinates; it is in contrast to the laboratory set of reference, in which spatial Fourier harmonics are coupled due to velocity shear.

Kinetic effects, such as finite Larmor radius effects, Landau and cyclotron damping and the numerous resulting kinetic instabilities, which are naturally not involved in the fluid description of plasma shear flows, require the development of a non-modal kinetic description of plasma shear flows. Note also, that because of the shearing of perturbations in shear flow, the component of the wave number along the direction of the velocity shear experiences secular growth with time[14],[16], and therefore results obtained on the base of the fluid description are valid only for finite times at which the condition $k_\perp(t)\,\rho_i \ll 1$, ($k_\perp$ is the component of the wave number across the magnetic field and ρ_i is thermal ion Larmor radius) for initially long wavelength perturbations with $k_\perp(t = t_0)\,\rho_i \ll 1$ holds. All this requires the development of the non-modal kinetic theory, which can properly describe the long-time evolution of the perturbations in shear flow. Motivated by these requirements, we have undertaken to work out the kinetic theory of plasma shear flows, which has Kelvin's method of shearing modes, or the so-called non-modal approach, as its foundation. The present paper focuses on the development of basic equations for linear and renormalized nonlinear non-modal kinetic theory for electrostatic perturbations of shear flow with homogeneous velocity shear and their solutions for kinetic drift instability of plasma shear flow. As a first attempt of such analytical investigation, we consider here the case of the homogeneous shearless magnetic field. In Section II, the governing linear integral equation for electrostatic potential is derived. In Section III, the solution of that equation for drift kinetic instability, which displays the non-modal evolution with time of the electrostatic potential in shear flow, is obtained. The nonlinear integral equation for electrostatic potential, which accounted for the scattering of ions by shearing modes with random phases in plasma shear flow, is obtained in Section IV. The solution of that equation, which describes the suppression of kinetic drift

instability in shear flow is obtained in Section V. A summary of the work is given in Conclusions, Section IV.

II. VLASOV–POISSON SYSTEM OF EQUATIONS IN SHEARED COORDINATES

Our theory is based on the Vlasov equations,

$$\frac{\partial F_\alpha}{\partial t}+\mathbf{v}\frac{\partial F_\alpha}{\partial \mathbf{r}}+\frac{e_\alpha}{m_\alpha}\left(\mathbf{E}_0\left(\mathbf{r}\right)+\frac{1}{c}\left[\mathbf{v}\times\mathbf{B}\right]-\bigtriangledown\varphi\left(\mathbf{r},t\right)\right)\frac{\partial F_\alpha}{\partial \mathbf{v}}=0, \tag{1}$$

for velocity distribution function F_α of α species ($\alpha=i$ for ions and $\alpha=e$ for electrons) in inhomogeneous electric field $\mathbf{E}_0\left(\mathbf{r}\right)$, directed across the homogeneous magnetic field $\mathbf{B}$, and Poisson equation for the perturbed electrostatic potential $\varphi\left(\mathbf{r},t\right)$,

$$\triangle\,\varphi\left(\mathbf{r},t\right)=-4\pi\sum_{\alpha=i,e}e_\alpha\int f_\alpha\left(\mathbf{v},\mathbf{r},t\right)d\mathbf{v}_\alpha, \tag{2}$$

where f_α is the perturbation of the equilibrium distribution function $F_{0\alpha}$, $F_\alpha=F_{0\alpha}+f_\alpha$. The kinetic theory for shear flows is developed as a rule[4, 11] in a frame that is shifted by $\mathbf{V}_0\left(x\right)$ in velocity space, but is unchanged in configuration space, leaving the inhomogeneous convective terms in Vlasov equation. The starting point of the derivation of the basic equations of the nonmodal kinetic theory is the transformation of Vlasov-Poisson system to convective (sheared) coordinates in velocity and configuration spaces.

We transform in Eq.(1) the variables $(t,\mathbf{r},\mathbf{v})$ onto new spatial variables $(t_c,\mathbf{r}_\alpha,\mathbf{v}_\alpha)$, connected by the relations

$$t=t_c,\qquad \mathbf{v}=\mathbf{v}_\alpha+\mathbf{U}_\alpha\left(\mathbf{r}_\alpha,t_c\right),\qquad \mathbf{r}=\mathbf{r}_\alpha+\int\limits_{t_{(0)}}^{t_c}\mathbf{U}_\alpha\left(\mathbf{r}_\alpha,t_{1c}\right)dt_{1c}, \tag{3}$$

or

$$\mathbf{v}_\alpha=\mathbf{v}-\mathbf{V}_\alpha\left(\mathbf{r}_\alpha,t\right),\qquad \mathbf{r}_\alpha=\mathbf{r}-\int\limits_{t_{(0)}}^{t}\mathbf{V}_\alpha\left(\mathbf{r},t_1\right)dt_1, \tag{4}$$

with set of reference moving with velocity $\mathbf{V}_\alpha\left(\mathbf{r},t\right)=\mathbf{U}_\alpha\left(\mathbf{r}_\alpha,t_c\right)$. In Eqs.(3),(4) $t_{(0)}$ denotes the time at which shear flow emerges. For the transformation of Eq.(1) to new variables we use the relations, which follows from Eqs.(3),(4),

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_c}-\frac{\partial V_{i\alpha}\left(\mathbf{r},t\right)}{\partial t}\frac{\partial}{\partial v_{i\alpha}}-V_{i\alpha}\left(\mathbf{r},t\right)\frac{\partial}{\partial r_{i\alpha}}, \tag{5}$$

$$\frac{\partial}{\partial v_i}=\frac{\partial}{\partial v_{i\alpha}},\qquad \frac{\partial v_{i\alpha}}{\partial r_j}=-\frac{\partial V_{i\alpha}}{\partial r_j},\qquad \frac{\partial r_{i\alpha}}{\partial r_j}=\delta_{ij}-\int\limits_{t_{(0)}}^{t}\frac{\partial V_{i\alpha}\left(\mathbf{r},t_1\right)}{\partial r_j}dt_1, \tag{6}$$

With these relations Eq.(1) takes the form

$$\frac{\partial F\left(t_c,\mathbf{r}_\alpha,\mathbf{v}_\alpha\right)}{\partial t_c}+v_{i\alpha}\frac{\partial F\left(t_c,\mathbf{r}_\alpha,\mathbf{v}_\alpha\right)}{\partial r_{i\alpha}}-\left(v_{i\alpha}+U_{i\alpha}\left(\mathbf{r}_\alpha,t_c\right)\right)\int\limits_{t_{(0)}}^{t}\frac{\partial V_{j\alpha}\left(\mathbf{r},t_1\right)}{\partial r_i}dt_1\frac{\partial F\left(t_c,\mathbf{r}_\alpha,\mathbf{v}_\alpha\right)}{\partial r_{j\alpha}}$$
$$-v_{i\alpha}\frac{\partial V_{j\alpha}\left(\mathbf{r},t_1\right)}{\partial r_i}\frac{\partial F\left(t_c,\mathbf{r}_\alpha,\mathbf{v}_\alpha\right)}{\partial v_{j\alpha}}+\frac{e_\alpha}{m_\alpha c}\left[\mathbf{v}_\alpha\times\mathbf{B}\right]\frac{\partial F_\alpha}{\partial\mathbf{v}_\alpha}$$
$$-\left\{\left[\frac{dV_{i\alpha}}{dt}-\frac{e_\alpha}{m_\alpha}\left(E_{0i}\left(\mathbf{r}\right)+\frac{1}{c}\left[\mathbf{V}_\alpha\times\mathbf{B}\right]_i\right)\right]-\frac{e_\alpha}{m_\alpha}\frac{\partial\varphi\left(\mathbf{r},t\right)}{\partial r_i}\right\}\frac{\partial F\left(t_c,\mathbf{r}_\alpha,\mathbf{v}_\alpha\right)}{\partial v_{i\alpha}}=0, \qquad (7)$$

where $dV_{i\alpha}/dt=\partial V_{i\alpha}/\partial t+V_{j\alpha}\left(\partial V_{i\alpha}/\partial r_j\right)$. When the velocity $V_{i\alpha}\left(\mathbf{r},t\right)$ is such as it vanishes the square brackets on last line of Eq.(7), Vlasov equation for shear flows, for which $U_{i\alpha}\left(\mathbf{r}_\alpha,t_c\right)\partial V_{j\alpha}\left(\mathbf{r},t\right)/\partial r_i\equiv 0$, contains only velocity shear parameter, V'_α, (instead of $V_\alpha(\mathbf{r})$ with laboratory frame variables),

$$\frac{\partial F_\alpha}{\partial t}+v_{\alpha x}\frac{\partial F_\alpha}{\partial x}-\left(v_{\alpha y}-v_{\alpha x}V'_\alpha t\right)\frac{\partial F_\alpha}{\partial y}+\omega_{c\alpha}v_{\alpha y}\frac{\partial F_\alpha}{\partial v_{\alpha x}}-\left(\omega_{c\alpha}+V'_\alpha\right)v_{\alpha x}\frac{\partial F_\alpha}{\partial v_{\alpha y}}$$
$$-\frac{e_\alpha}{m_\alpha}\left(\frac{\partial\varphi}{\partial x}-V'_\alpha t\frac{\partial\varphi}{\partial y}\right)\frac{\partial F_\alpha}{\partial v_{\alpha x}}+v_{\alpha z}\frac{\partial F_\alpha}{\partial z_\alpha}-\frac{e_\alpha}{m_\alpha}\frac{\partial\varphi}{\partial y}\frac{\partial F_\alpha}{\partial v_{\alpha y}}-\frac{e_\alpha}{m_\alpha}\frac{\partial\varphi}{\partial z_\alpha}\frac{\partial F_\alpha}{\partial v_{\alpha z}}=0. \qquad (8)$$

In this paper we consider stationary plasma shear flows across the magnetic field with homogeneous velocity shear, for which

$$\mathbf{V}_\alpha=\mathbf{V}_0=-\frac{c}{B}\frac{dE_0}{dx}x\mathbf{e}_y=\frac{dV_0}{dx}x\mathbf{e}_y, \qquad (9)$$

with $\mathbf{E}_0\left(\mathbf{r}\right)=\left(dE_0/dx\right)x\mathbf{e}_x$ and $dE_0/dx=const$. For this case the transformations (3) have a form

$$t=t_c,\qquad v_x=v_{\alpha x},\qquad v_y=v_{\alpha y}+V'_0x_\alpha,\qquad v_z=v_{z\alpha}, \qquad (10)$$
$$x=x_\alpha,\qquad y=y_\alpha+V'_0x_\alpha t_c,\qquad z=z_\alpha \qquad (11)$$

where $V'_0=dV_0/dx$, and $t_{(0)}=0$ was assumed.

With leading center coordinates,

$$X_\alpha=x_\alpha+\frac{v_\perp}{\sqrt{\mu_\alpha}\omega_{c\alpha}}\sin\left(\phi_1-\sqrt{\mu_\alpha}\omega_{c\alpha}t\right),$$
$$Y_\alpha=y_\alpha-\frac{v_\perp}{\mu_\alpha\omega_{c\alpha}}\cos\left(\phi_1-\sqrt{\mu_\alpha}\omega_{c\alpha}t\right)-V'_0t\ \left(X_\alpha-x_\alpha\right),\qquad z_1=z-v_zt, \qquad (12)$$

and velocity space coordinates (see, also Ref.[19]),

$$v_{\alpha x}=v_\perp\cos\phi,\qquad v_{\alpha y}=\sqrt{\mu_\alpha}v_\perp\sin\phi,\qquad \phi=\phi_1-\sqrt{\mu_\alpha}\omega_{c\alpha}t,\qquad v_z=v_{\alpha z} \qquad (13)$$

where $\mu_\alpha=1+V'_0/\omega_{c\alpha}>0$, the equation for the perturbation f_α of the equilibrium distribution $F_{0\alpha}$ function ($F_\alpha=F_{0\alpha}+f_\alpha$) has a simple form,

$$\frac{\partial f_\alpha}{\partial t}+\frac{e_\alpha}{m_\alpha\sqrt{\mu_\alpha}\omega_{c\alpha}}\left(\frac{\partial\varphi}{\partial X_\alpha}\frac{\partial f_\alpha}{\partial Y_\alpha}-\frac{\partial\varphi}{\partial Y_\alpha}\frac{\partial f_\alpha}{\partial X_\alpha}\right)+\frac{e_\alpha}{m_\alpha}\frac{\sqrt{\mu_\alpha}\omega_{c\alpha}}{v_\perp}\left(\frac{\partial\varphi}{\partial\phi_1}\frac{\partial f_\alpha}{\partial v_\perp}-\frac{\partial\varphi}{\partial v_\perp}\frac{\partial f_\alpha}{\partial\phi_1}\right)$$
$$-\frac{e_\alpha}{m_\alpha}\frac{\partial\varphi}{\partial z_\alpha}\frac{\partial f_\alpha}{\partial v_{z\alpha}}=\frac{e_\alpha}{m_\alpha}\left[\frac{1}{\sqrt{\mu_\alpha}\omega_{c\alpha}}\frac{\partial\varphi}{\partial Y_\alpha}\frac{\partial F_{0\alpha}}{\partial X_\alpha}-\frac{\sqrt{\mu_\alpha}\omega_{c\alpha}}{v_\perp}\frac{\partial\varphi}{\partial\phi_\alpha}\frac{\partial F_{0\alpha}}{\partial v_{\perp\alpha}}+\frac{\partial\varphi}{\partial z_\alpha}\frac{\partial F_{0\alpha}}{\partial v_{z\alpha}}\right]. \qquad (14)$$

We assume that $|V_0'/\omega_{c\alpha}| \ll 1$ and put in what follows $\mu_\alpha = 1$. It is interesting to note, that the equilibrium distribution function $F_{0\alpha}$, which in laboratory frame contains the spatial inhomogeneity resulted from electric field $\mathbf{E}_0(\mathbf{r})$, does not contain such inhomogeneity in convective coordinates (see Appendix 1). In what follows we consider the equilibrium distribution function F_{i0} as a Maxwellian,

$$F_{0\alpha} = \frac{n_{0\alpha}(X_\alpha)}{\left(2\pi v_{T\alpha}^2\right)^{3/2}} \exp\left(-\frac{v_\perp^2 + v_z^2}{v_{T\alpha}^2}\right). \tag{15}$$

assuming the inhomogeneity of the density of plasma shear flow species on coordinate X_α. It follows from Eq.(8), as well as from Eq.(14), that for $V_0' = const$, these equations do not contain the spatial inhomogeneities, originated from inhomogeneity of the flow velocity $V_0(x)$. Therefore, the spatially homogeneous, but time dependent, Eqs.(8) and (14) may be Fourier transformed over the variables $x_\alpha, y_\alpha, z_\alpha$ with conjugate wave numbers k_x, k_y and k_z and the temporal evolution of the separate spatial Fourier mode of the perturbations of the distribution function, f_α and of the electrostatic potential, $\varphi(\mathbf{k}, t)$ may be traced upon. On this way we present the potential $\varphi(\mathbf{r}, t)$ in form

$$\begin{aligned}\varphi(x_\alpha, y_\alpha, z_\alpha, t) &= \int \varphi(k_x, k_y, k_z, t)\, e^{ik_x x_\alpha + ik_y y_\alpha + ik_z z_\alpha} dk_x dk_y dk_z \\ &= \int \varphi(k_x, k_y, k_z, t) \exp\Big[ik_x X_\alpha + ik_y Y_\alpha + ik_z z_\alpha \\ &\quad - i\frac{k_\perp(t)\, v_\perp}{\omega_{c\alpha}} \sin(\phi - \omega_{c\alpha} t - \theta(t))\Big] dk_x dk_y dk_z,\end{aligned} \tag{16}$$

where

$$k_\perp^2(t) = \left(k_x - V_0' t k_y\right)^2 + k_y^2 \tag{17}$$

and $\tan\theta = k_y/(k_x - V_0' t k_y)$. The solution to linearized Eq.(14) is calculated easily for any values of the velocity shear rate V_0' and it is equal to

$$\begin{aligned}f_\alpha(t, k_x, k_y, k_z, v_\perp, \phi, v_z, z_1) &= \frac{ie_\alpha}{m_\alpha} \sum_{n=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \int^{t} dt_1 \varphi(t_1, k_x, k_y, k_z) \\ \times \exp\Big(-ik_z v_z(t - t_1) &+ in(\phi_1 - \omega_c t - \theta(t)) - in_1(\phi_1 - \omega_c t_1 - \theta(t))\Big) \\ \times J_n\left(\frac{k_\perp(t)\, v_\perp}{\omega_c}\right) J_{n_1}&\left(\frac{k_\perp(t_1)\, v_\perp}{\omega_c}\right)\left[\frac{k_y}{\omega_{c\alpha}}\frac{\partial F_\alpha}{\partial X_\alpha} + \frac{\omega_c n_1}{v_\perp}\frac{\partial F_\alpha}{\partial v_\perp} + k_{1z}\frac{\partial F_\alpha}{\partial v_z}\right] \\ &+ f_\alpha(t = t_0, k_x, k_y, k_z, v_\perp \phi, v_z).\end{aligned} \tag{18}$$

The Poisson equation for separate spatial Fourier harmonic $\varphi(\mathbf{k}, t)$ gives the governing integral equation[17]

$$\begin{aligned}\Big[\left(k_x - V_0' t k_y\right)^2 + k_y^2 + k_z^2\Big]\varphi(\mathbf{k}, t) &= \sum_{\alpha=i,e} \frac{i}{\lambda_{D\alpha}^2} \sum_{n=-\infty}^{\infty} \int_{t_0}^{t} dt_1 \varphi(\mathbf{k}, t_1) \\ \times I_n\left(k_\perp(t)\, k_\perp(t_1)\, \rho_\alpha^2\right) e^{-\frac{1}{2}\rho_\alpha^2\left(k_\perp^2(t) + k_\perp^2(t_1)\right)} &e^{-\frac{1}{2}k_z^2 v_{T\alpha}^2 (t - t_1)^2 - in\omega_{c\alpha}(t - t_1) - in(\theta(t) - \theta(t_1))} \\ \times \left[k_y v_{d\alpha} - n\omega_{c\alpha} + ik_z^2 v_{T\alpha}^2 (t - t_1)\right] &- 4\pi \sum_{\alpha=i,e} e_\alpha \delta n_\alpha(\mathbf{k}, t, t_0),\end{aligned} \tag{19}$$

where

$$
\begin{aligned}
&4\pi\sum_{\alpha=i,e}e_{\alpha}\delta n_{\alpha}\left(\mathbf{k},t,t_{0}\right)\\
&=8\pi^{2}\sum_{\alpha=i,e}e_{\alpha}\int\limits_{-\infty}^{\infty}dv_{z}e^{-ik_{z}v_{z}t}\int\limits_{0}^{\infty}dv_{\perp}v_{\perp}J_{0}\left(\frac{k_{\perp}\left(t\right)v_{\perp}}{\omega_{c\alpha}}\right)f_{\alpha}\left(t=t_{0},\mathbf{k},v_{\perp},v_{z}\right),
\end{aligned}
\tag{20}
$$

and where $f_{\alpha}\left(t=t_{0},\mathbf{k},v_{\perp},v_{z}\right)$ is the initial, determined at $t=t_{0}$ perturbation, assumed here as not dependent on ϕ, of the distribution function F_{α}. It follows from Eq.(19), that initial perturbation $\varphi\left(\mathbf{k},t=t_{0}\right)$ of the self-consistent electrostatic potential is equal to

$$
\varphi\left(\mathbf{k},t=t_{0}\right)=-\frac{4\pi}{k_{\perp}^{2}\left(t_{0}\right)+k_{z}^{2}}\sum_{\alpha=i,e}e_{\alpha}\delta n_{\alpha}\left(\mathbf{k},t_{0},t_{0}\right).
\tag{21}
$$

Eq.(19) presents minimal model of the linear kinetic theory of plasma shear flows, which incorporates the shear flow effects. In Eq.(19) these effects are concentrated in the time dependent arguments of the Bessel function and manifest itself as a time-dependent finite-Larmor-radius effects.

The integration by parts of the first term on the right of Eq.(19) gives the integral equation, which appears to be more convenient and transparent for further analysis,

$$
\begin{aligned}
&\left[\left(k_{x}-V_{0}'tk_{y}\right)^{2}+k_{y}^{2}+k_{z}^{2}\right]\varphi\left(\mathbf{k},t\right)=-\sum_{\alpha=i,e}\frac{1}{\lambda_{D\alpha}^{2}}\varphi\left(\mathbf{k},t\right)+\sum_{\alpha=i,e}\frac{1}{\lambda_{D\alpha}^{2}}\sum_{n=-\infty}^{\infty}\int\limits_{t_{0}}^{t}dt_{1}\frac{d}{dt_{1}}\Big\{\varphi\left(\mathbf{k},t_{1}\right)\\
&\times I_{n}\left(k_{\perp}\left(t\right)k_{\perp}\left(t_{1}\right)\rho_{\alpha}^{2}\right)e^{-\frac{1}{2}\rho_{\alpha}^{2}\left(k_{\perp}^{2}(t)+k_{\perp}^{2}(t_{1})\right)-in(\theta(t)-\theta(t_{1}))}\Big\}e^{-\frac{1}{2}k_{z}^{2}v_{T\alpha}^{2}(t-t_{1})^{2}-in\omega_{c\alpha}(t-t_{1})}\\
&+\sum_{\alpha=i,e}\frac{i}{\lambda_{D\alpha}^{2}}\sum_{n=-\infty}^{\infty}\int\limits_{t_{0}}^{t}dt_{1}\varphi\left(\mathbf{k},t_{1}\right)k_{y}v_{d\alpha}\,I_{n}\left(k_{\perp}\left(t\right)k_{\perp}\left(t_{1}\right)\rho_{\alpha}^{2}\right)\\
&\times\exp\left[-\frac{1}{2}\rho_{\alpha}^{2}\left(k_{\perp}^{2}\left(t\right)+k_{\perp}^{2}\left(t_{1}\right)\right)-in\left(\theta\left(t\right)-\theta\left(t_{1}\right)\right)-\frac{1}{2}k_{z}^{2}v_{T\alpha}^{2}\left(t-t_{1}\right)^{2}-in\omega_{c\alpha}\left(t-t_{1}\right)\right]\\
&-4\pi\sum_{\alpha=i,e}e_{\alpha}\delta n_{\alpha}\left(\mathbf{k},t,t_{0}\right)+\sum_{\alpha=i,e}\frac{1}{\lambda_{D\alpha}^{2}}\varphi\left(\mathbf{k},t_{0}\right)P_{\alpha}\left(t,t_{0}\right),
\end{aligned}
\tag{22}
$$

where $v_{d\alpha}=cT_{\alpha}/eBL_{n}$ is the diamagnetic drift velocity, $L_{n}^{-1}=-d\ln n_{0\alpha}\left(x\right)/dx$, ρ_{α} is thermal Larmor radius, and

$$
\begin{aligned}
&P_{\alpha}\left(t,t_{0}\right)=\sum_{n=-\infty}^{\infty}I_{n}\left(k_{\perp}\left(t\right)k_{\perp}\left(t_{0}\right)\rho_{\alpha}^{2}\right)\\
&\times\exp\left[-\frac{1}{2}\rho_{\alpha}^{2}\left(k_{\perp}^{2}\left(t\right)+k_{\perp}^{2}\left(t_{0}\right)\right)-in\left(\theta\left(t\right)-\theta\left(t_{0}\right)\right)-\frac{1}{2}k_{z}^{2}v_{T\alpha}^{2}\left(t-t_{0}\right)^{2}-in\omega_{c\alpha}\left(t-t_{0}\right)\right].
\end{aligned}
\tag{23}
$$

Note, that $P_{\alpha}\left(t_{0},t_{0}\right)=1$.

It is important to note the alternative, explicitly causal representation of Eq.(22) with function $\Phi\left(\mathbf{k},t\right)=\varphi\left(\mathbf{k},t\right)\Theta\left(t-t_{0}\right)$, where $\Theta\left(t-t_{0}\right)$ is the unit-step Heaviside function (it is equal to zero

for $t < t_0$ and equal to unity for $t \geq t_0$). That equation for $\Phi\left(\mathbf{k},t\right)$ has a form

$$\left[\left(k_x - V_0' t k_y\right)^2 + k_y^2 + k_z^2\right]\Phi\left(\mathbf{k},t\right) + 4\pi\sum_{\alpha=i,e} e_\alpha \delta n_\alpha\left(\mathbf{k},t,t_0\right)$$

$$= -\sum_{\alpha=i,e}\frac{1}{\lambda_{D\alpha}^2}\int\limits_{t_0}^{t} dt_1\frac{d}{dt_1}\Phi\left(\mathbf{k},t\right) + \sum_{\alpha=i,e}\frac{1}{\lambda_{D\alpha}^2}\sum_{n=-\infty}^{\infty}\int\limits_{t_0}^{t} dt_1\,\frac{d}{dt_1}\Big\{\,\Phi\left(\mathbf{k},t_1\right)$$

$$\times I_n\left(k_\perp\left(t\right)k_\perp\left(t_1\right)\rho_\alpha^2\right)e^{-\frac{1}{2}\rho_\alpha^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)-in(\theta(t)-\theta(t_1))}\Big\}\,e^{-\frac{1}{2}k_z^2 v_{T\alpha}^2(t-t_1)^2-in\omega_{c\alpha}(t-t_1)}$$

$$+\sum_{\alpha=i,e}\frac{i}{\lambda_{D\alpha}^2}\sum_{n=-\infty}^{\infty}\int\limits_{t_0}^{t} dt_1\Phi\left(\mathbf{k},t_1\right)k_y v_{d\alpha} I_n\left(k_\perp\left(t\right)k_\perp\left(t_1\right)\rho_\alpha^2\right)$$

$$\times\exp\left[-\frac{1}{2}\rho_\alpha^2\left(k_\perp^2\left(t\right)+k_\perp^2\left(t_1\right)\right) - in\left(\theta\left(t\right)-\theta\left(t_1\right)\right) - \frac{1}{2}k_z^2 v_{T\alpha}^2\left(t-t_1\right)^2 - in\omega_{c\alpha}\left(t-t_1\right)\right]. \tag{24}$$

Using the quasi neutrality approximation with $\left(k_\perp^2\left(t\right)+k_z^2\right)\lambda_{D\alpha}^2 \ll 1$, and averaging this equation over the time $t \gg \omega_{ci}^{-1}$, we obtain from Eq.(24) the equation, which is relevant for the analysis of the low frequency drift type perturbations,

$$\int\limits_{t_0}^{t} dt_1\frac{d}{dt_1}\left\{\Phi\left(\mathbf{k},t_1\right)\left[\left(1+\tau\right) - I_0\left(k_\perp\left(t\right)k_\perp\left(t_1\right)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}\right]\right\}$$

$$-i\int\limits_{t_0}^{t} dt_1\Phi\left(\mathbf{k},t_1\right)k_y v_{di} I_0\left(k_\perp\left(t\right)k_\perp\left(t_1\right)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}$$

$$= -\int\limits_{t_0}^{t} dt_1\frac{d}{dt_1}\left(\Phi\left(\mathbf{k},t_1\right)I_0\left(k_\perp\left(t\right)k_\perp\left(t_1\right)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}\right)\left(1-e^{-\frac{1}{2}k_z^2 v_{Ti}^2(t-t_1)^2}\right)$$

$$-i\int\limits_{t_0}^{t} dt_1\Phi\left(\mathbf{k},t_1\right)k_y v_{di} I_0\left(k_\perp\left(t\right)k_\perp\left(t_1\right)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}\left(1-e^{-\frac{1}{2}k_z^2 v_{Ti}^2(t-t_1)^2}\right)$$

$$+\tau\int\limits_{t_0}^{t} dt_1\left(\frac{d\Phi\left(\mathbf{k},t_1\right)}{dt_1} + ik_y v_{de}\Phi\left(\mathbf{k},t_1\right)\right)e^{-\frac{1}{2}k_z^2 v_{Te}^2(t-t_1)^2}, \tag{25}$$

where $\tau = T_i/T_e$. Eq.(25) introduces obvious time scales which determine different stages of the temporal evolution of the electrostatic potential in plasma shear flows. As it follows from Eq.(17), the time dependence of $k_\perp\left(t\right)$ is negligible in times $t \ll \left(V_0'\right)^{-1}$ and may be neglected; on that stage Eq.(25) determines the ordinary modal evolution of perturbations as in plasma without shear flow. In times $\left(V_0'\right)^{-1} \ll t \ll t_s = \left(V_0' k_y \rho_i\right)^{-1}$ time dependence of $k_\perp$ gradually enhances the non-modal evolution of the potential with time. In times $t \gg t_s$, non-modality becomes the dominant effect of the temporal evolution of the potential $\Phi\left(\mathbf{k},t\right)$. In next Section we will obtain by successive approximations the approximate solution to Eq.(25) for long wavelength, $k_\perp\left(t_0\right)\rho_i < 1$, perturbations with weak ion Landau damping, for which $\left|1-e^{-\frac{1}{2}k_z^2 v_{Ti}^2(t-t_0)^2}\right| \ll 1$. The solution will be obtained

without application the spectral transformation over time for the case of a weak velocity shear, or a small time, for which condition $|V_0'| \leq t \ll t_s$ is met, and for long times, $t \gg t_s$.

III.LINEAR NON-MODAL EVOLUTION OF THE KINETIC DRIFT INSTABILITY OF PLASMA SHEAR FLOW

If $k_\perp(t)\rho_i < 1$ at time $t = t_{(0)} = 0$ at which the shear flow emerge , i.e. $V_0' = V_0'\Theta(t)$, we will get $k_\perp(t)\rho_i < 1$ on times $t < t_s$ throughout. By using the approximation

$$I_0\left(k_\perp(t)\,k_\perp(t_1)\,\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}$$
$$\approx b_i + \left(k_x k_y V_0'\rho_i^2(t+t_1) - \frac{1}{2}k_y^2\rho_i^2(V_0')^2\left(t^2+t_1^2\right)\right)\Theta(t) \tag{26}$$

in Eq.(25), we present Eq.(25) in the form

$$\int\limits_{t_0}^{t} dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1} + i\omega(\mathbf{k})\,\Phi(\mathbf{k},t_1)\right)$$
$$= -\frac{b_i}{a_i}\int\limits_{t_0}^{t} dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1} + ik_y v_{di}\Phi(\mathbf{k},t_1)\right)\left(1 - e^{-\frac{1}{2}k_z^2 v_{Ti}^2(t-t_0)^2}\right)$$
$$+\frac{b_i}{a_i}\int\limits_{0}^{t} dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1} + ik_y v_{di}\Phi(\mathbf{k},t_1)\right)\left(\frac{k_x}{k_y}\frac{(t+t_1)}{a_i V_0' t_s^2} - \frac{(t^2+t_1^2)}{2a_i t_s^2}\right)$$
$$+\int\limits_{0}^{t} dt_1\Phi(\mathbf{k},t_1)\,\frac{1}{a_i V_0' t_s^2}\left(\frac{k_x}{k_y} - V_0' t\right)$$
$$-\frac{b_i}{a_i}\int\limits_{0}^{t} dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1} + ik_y v_{di}\Phi(\mathbf{k},t_1)\right)\left(\frac{k_x}{k_y}\frac{(t+t_1)}{a_i V_0' t_s^2} - \frac{(t^2+t_1^2)}{2a_i t_s^2}\right)\left(1 - e^{-\frac{1}{2}k_z^2 v_{Ti}^2(t-t_0)^2}\right)$$
$$+\frac{\tau}{a_i}\int\limits_{t_0}^{t} dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1} + ik_y v_{de}\Phi(\mathbf{k},t_1)\right)\left(1 - e^{-\frac{1}{2}k_z^2 v_{Te}^2(t-t_0)^2}\right), \tag{27}$$

where $b_i = 1 - k_\perp^2\rho_i^2$, $a_i = \tau + k_\perp^2\rho_i^2$ and

$$\omega(\mathbf{k}) = -\frac{b_i}{a_i}k_y v_{di}. \tag{28}$$

The first term in the right side of Eq.(27) determines the ion Landau damping; this term is the same as in plasma without shear flow. The next three terms originated from shear flow and determine the corrections to the frequency and ion Landau damping provided by shear flow. The right side of

Eq.(27) is small for $(V')^{-1} < t < t_s$, $\tau < 1$ and for weak ion Landau damping. Therefore the solution to Eq.(27) we seek in the form

$$\Phi(\mathbf{k},t) = C\exp\left(-i\omega(\mathbf{k})t + i\nu(\mathbf{k},t)\right). \tag{29}$$

Inserting Eq.(29) into Eq.(27) and neglecting the derivative $d\nu(\mathbf{k},t)/dt$ in the right side of Eq.(27), we obtain for $\nu(\mathbf{k},t)$ following equation, assuming that $t_0 \to -\infty$,

$$\int\limits_{t_0\to-\infty}^{t} dt_1 \Phi(\mathbf{k},t_1)\left[i\frac{d\nu(\mathbf{k},t_1)}{dt_1} - i\delta\omega(\mathbf{k}) - \frac{1}{a_i t_s^2}\left(i\omega(\mathbf{k})t_1^2\left(1+\frac{a_i}{b_i}\right) - t_1\right)\right] = 0, \tag{30}$$

where

$$\delta\omega(\mathbf{k}) = i\frac{\omega(\mathbf{k})(\omega(\mathbf{k}) - k_y v_{di})}{k_z v_{Ti}}\frac{b_i}{a_i}\sqrt{\frac{\pi}{2}}W\left(\frac{\omega(\mathbf{k})}{\sqrt{2}k_z v_{Ti}}\right)$$
$$+ i\tau\frac{\omega(\mathbf{k})(\omega(\mathbf{k}) - k_y v_{de})}{a_i k_z v_{Te}}\sqrt{\frac{\pi}{2}}W\left(\frac{\omega(\mathbf{k})}{\sqrt{2}k_z v_{Te}}\right), \tag{31}$$

with $\omega(\mathbf{k})$ defined by Eq.(28) and $W(z) = \exp(-z^2)\left(1 + \frac{2i}{\sqrt{\pi}}\int\limits_0^z d\zeta e^{\zeta^2}\right)$. It is interesting to note, that $\mathrm{Im}\delta\omega(\mathbf{k}) = \gamma(\mathbf{k})$ is right well known growth rate of the kinetic drift instability [18] determined in plasma theory approximately as $\gamma(\mathbf{k}) = -\mathrm{Im}\varepsilon(\mathbf{k},\omega(\mathbf{k}))/(\partial\mathrm{Re}\varepsilon(\mathbf{k},\omega(\mathbf{k}))/\partial\omega(\mathbf{k}))$, where $\varepsilon(\mathbf{k},\omega)$ is ordinary electrostatic dielectric permittivity of the plasma in magnetic field. By equation to zero the expression in square brackets in Eq.(30), we obtain simple solution for $\nu(\mathbf{k},t)$, which determines the non-modal evolution of the potential Φ with time,

$$\Phi(\mathbf{k},t) = \Phi_0\exp\left[-i\omega(\mathbf{k})t\left(1 - \frac{1+\tau}{a_i b_i}\frac{t^2}{3t_s^2}\right) + iRe\delta\omega(\mathbf{k})t + \left(\gamma(\mathbf{k}) - \frac{t}{2a_i t_s^2}\right)t\right]. \tag{32}$$

As it follows from Eq.(32), non-modal effects, which reveal in non-modal reduction of the frequency and growth rate, are negligible at $t \ll t_s$ and become dominant at $t \sim t_s$. Note, that for $\tau \gg k_\perp^2\rho_i^2$ the time $a_i^{1/2}t_s$ is approximately equal to time $t_2 = (V_0' k_\perp \rho_s)^{-1}$ of the transition to strongly non-modal regime in the fluid theory of the drift turbulence of the plasma shear flow[16].

In the laboratory frame of reference spatial Fourier mode (32) is observed as sheared mode with time dependent component of the wave number $k_{x(lab)} = k_x - k_y V_0' t$ directed along the velocity shear, and therefore is quite different from the normal mode assumption,

$$\Phi(\mathbf{r},t) = \int dk_x \int dk_y \int dk_z \Phi_0 \exp\left(ik_x x + ik_y y + ik_z z - ik_y V_0' t x\right)$$
$$\times\exp\left[-i\omega(\mathbf{k})t\left(1 - \frac{1+\tau}{a_i b_i}\frac{t^2}{3t_s^2}\right) + iRe\delta\omega(\mathbf{k})t + \left(\gamma(\mathbf{k}) - \frac{t}{2a_i t_s^2}\right)t\right]. \tag{33}$$

The mode shearing, which is other non-modal effect, becomes pronounced in times $t \geq \gamma^{-1}$ when $V_0' \simeq \gamma$. Only in times, for which $|V_0' t| \ll 1$ solution (33) has a normal mode form, $\Phi(\mathbf{k},t) \sim \exp(ik_x x + ik_y y + ik_z z - i\omega(\mathbf{k})t)$. Because of the time dependence $k_{x(lab)} = k_x - k_y V_0' t$ of the wave

number component along the flow shear, the modes in the laboratory frame in times $t \gg (V_0')^{-1}$ become increasingly one–dimensional zonal–like as the perturbed $E \times B$ velocity tilts more and more closely parallel to y-axis.

The exceptional advantage of the non-modal approach is a possibility to perform the analysis of a plasma evolution on any finite time domain. Now we consider the temporal evolution of drift perturbations at times $t > t_0 \gg t_s$ on the base of Eq.(22) averaged over the time $t \gg \omega_{ci}^{-1}$. Note, that for the averaged Eq.(22) ,

$$P_i(t, t_0) = I_0\left(k_\perp(t) k_\perp(t_0) \rho_i^2\right) \exp\left[-\frac{1}{2}\rho_i^2\left(k_\perp^2(t) + k_\perp^2(t_0)\right) - \frac{1}{2}k_z^2 v_{Ti}^2 (t - t_0)^2\right], \tag{34}$$

and $P_i(t_0, t_0) = I_0\left(k_\perp^2(t_0)\rho_i^2\right) e^{-\rho_i^2 k_\perp^2(t_0)}$. We consider the times $t > t_0 \gg t_s$, for which $k_\perp(t)$ becomes large enough, so that $k_\perp(t)\rho_i > k_\perp(t_0)\rho_i > 1$ and $P_i(t_0, t_0) \simeq \left(\sqrt{2\pi} k_\perp(t_0)\rho_i\right)^{-1}$. The initial value $\varphi(\mathbf{k}, t = t_0)$ for the averaged potential in that case have to be corrected and is equal to

$$\varphi(\mathbf{k}, t = t_0) = \varphi(\mathbf{k}, t_0) \frac{t_s}{\sqrt{2\pi} t_0}, \tag{35}$$

where $\varphi(\mathbf{k}, t_0)$ is the initial value for Eq.(22) without averaging. For the times considered

$$I_0\left(k_\perp(t) k_\perp(t_1)\rho_i^2\right) e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t) + k_\perp^2(t_1)\right) - \frac{1}{2}k_z^2 v_{Ti}^2 (t - t_1)^2} \approx \frac{t_s}{\sqrt{2\pi t t_1}} \exp\left(-\frac{1}{2}\kappa_i^2 (t - t_1)^2\right), \tag{36}$$

where

$$\kappa_i^2 = \frac{1}{t_s^2} + k_z^2 v_{Ti}^2. \tag{37}$$

With this approximation, Eq.(22) becomes

$$\begin{aligned}
(1+\tau)\int_{t_0}^{t} dt_1 \frac{d\varphi(\mathbf{k}, t_1)}{dt_1} = \int_{t_0}^{t} dt_1 \frac{d}{dt_1}\left[\varphi(\mathbf{k}, t_1)\frac{t_s}{(2\pi t t_1)^{1/2}} e^{-\frac{1}{2t_s^2}(t-t_1)^2}\right] e^{-\frac{1}{2}k_z^2 v_{Ti}^2 (t-t_1)^2} \\
+ i k_y v_{di} \int_{t_0}^{t} dt_1 \varphi(\mathbf{k}, t_1) \frac{t_s}{(2\pi t t_1)^{1/2}} e^{-\frac{\kappa_i^2}{2}(t-t_1)^2} \\
+ \tau \int_{t_0}^{t} dt_1 \left(\frac{d\varphi(\mathbf{k}, t_1)}{dt_1} + i k_y v_{de}\varphi(\mathbf{k}, t_1)\right) e^{-\frac{1}{2}k_z^2 v_{Te}^2 (t-t_1)^2} \\
+ \varphi(\mathbf{k}, t_0) \frac{t_s}{(2\pi t t_0)^{1/2}} \left(e^{-\frac{\kappa_i^2}{2}(t-t_0)^2} - \left(\frac{t}{t_0}\right)^{1/2}\right).
\end{aligned} \tag{38}$$

In Eq.(38), electron term may be omitted, because for times $t - t_0 > t_s$ and $V_0' \sim \gamma = k_y v_{di} k_\perp^2 \rho_i^2$, where γ is the growth rate of the drift kinetic instability,

$$\exp\left(-\frac{1}{2}k_z^2 v_{Te}^2 (t - t_1)^2\right) < \exp\left(-\frac{1}{2}k_z^2 v_{Te}^2 t_s^2\right) \sim \exp\left(-\frac{k_z^2}{\tau k_y^2}\frac{m_i}{m_e}\frac{k_\perp^2 L_n^2}{(k_\perp \rho_i)^4}\right) \ll 1. \tag{39}$$

In zero approximation over t_s/t we have for $\varphi(\mathbf{k},t)$ equation $\int\limits_{t_0}^{t} dt_1 d\varphi(\mathbf{k},t_1)/dt_1 = 0$ with solution $\varphi(\mathbf{k},t_1)/dt_1 = \varphi_0 = const$. Accounting for in first approximation small right hand side of Eq.(38), we seek solution for $\varphi(\mathbf{k},t)$ in the form

$$\varphi(\mathbf{k},t) = \varphi_0 \exp(\nu(\mathbf{k},t)), \tag{40}$$

where $\nu(\mathbf{k},t) = O(t_s/t)$. For times $t \geq t_0 \gg \kappa^{-1}$ we can omit in Eq.(38) the terms with small exponents $\exp\left(-(\kappa_i^2/2)(t-t_0)^2\right)$ and obtain the following solution for $\varphi(\mathbf{k},t)$:

$$\varphi(\mathbf{k},t) = \varphi_0 \exp\left(\frac{1}{\sqrt{2\pi}\kappa_i^2 t_s t} + i\frac{k_y v_{di} t_s}{2\kappa_i t}\right), \tag{41}$$

which gradually becomes a zero-frequency cell-like perturbation.

IV.NONLINEAR PHASE SHIFT IN SHEAR FLOW

Nonlinear terms in the left hand side of Eq.(14) can drastically change linear non-modal solutions (33) and (41). Because of the nonlinearities, accounted in the left side of Eq.(14), variables $X, Y, v_\perp, \phi$ and z become coupled and determined by equations of characteristics,

$$\begin{aligned} dt = \frac{dX}{-\frac{e}{m_i\omega_{ci}}\frac{\partial\varphi}{\partial Y}} = \frac{dY}{\frac{e}{m_i\omega_{ci}}\frac{\partial\varphi}{\partial X}} = \frac{dv_\perp}{\frac{e}{m_i}\frac{\omega_{ci}}{v_\perp}\frac{\partial\varphi}{\partial\phi_1}} = \frac{d\phi_1}{-\frac{e}{m_i}\frac{\omega_{ci}}{v_\perp}\frac{\partial\varphi}{\partial v_\perp}} = \frac{dv_z}{-\frac{e}{m_i}\frac{\partial\varphi}{\partial z_1}} \\ = \frac{df_i}{\frac{e}{m_i\omega_{ci}}\frac{\partial\varphi}{\partial Y}\frac{\partial F_{i0}}{\partial X} - \frac{e}{m_i}\frac{\omega_{ci}}{v_\perp}\frac{\partial\varphi}{\partial\phi_1}\frac{\partial F_{i0}}{\partial v_\perp} + \frac{e}{m_i}\frac{\partial\varphi}{\partial z_1}\frac{\partial F_{i0}}{\partial v_z}}. \end{aligned} \tag{42}$$

Last equation in system (42) gives nonlinear solution for the perturbation of the ion distribution function f_i with known F_{i0},

$$f_i = \frac{e}{m}\int^{t}\left[\frac{1}{\omega_{ci}}\frac{\partial\varphi}{\partial Y}\frac{\partial F_{i0}}{\partial X} - \frac{\omega_{ci}}{v_\perp}\frac{\partial\varphi}{\partial\phi_1}\frac{\partial F_{i0}}{\partial v_\perp} + \frac{\partial\varphi}{\partial z_1}\frac{\partial F_{i0}}{\partial v_z}\right]dt', \tag{43}$$

in which coordinates X, Y, $v_\perp$, ϕ are $X = \bar{X} + \delta X$, $Y = \bar{Y} + \delta Y$, $\phi = \bar{\phi} + \delta\phi$, where $\bar{X}$, $\bar{Y}$, are the guiding center coordinates averaged over the turbulent pulsations, and $\delta X(t)$, $\delta Y(t)$, $\delta\phi$ are random ion orbit disturbances due to their scattering by electrostatic low frequency drift turbulence. The disturbances are assumed sufficiently small and, after the averaging over the times $t \gg (\omega_{ci})^{-1}$, they are determined by the equations

$$\delta X = -\frac{e}{m_i\omega_{ci}}\int\limits_{t_0}^{t}\frac{\partial\varphi}{\partial\bar{Y}}dt_1 = -\frac{c}{B}\int\limits_{t_0}^{t}dt_1\int d\mathbf{k}\varphi(\mathbf{k},t_1)k_y J_0\left(\frac{k_\perp(t_1)v_\perp}{\omega_{ci}}\right)e^{i\Psi}, \tag{44}$$

$$\delta Y = \frac{e}{m_i\omega_{ci}}\int\limits_{t_0}^{t}\frac{\partial\varphi}{\partial\bar{X}}dt_1 = \frac{c}{B}\int\limits_{t_0}^{t}dt_1\int d\mathbf{k}\varphi(\mathbf{k},t_1)k_x J_0\left(\frac{k_\perp(t_1)v_\perp}{\omega_{ci}}\right)e^{i\Psi}, \tag{45}$$

$$\delta\phi = -\frac{e}{m_i}\frac{\omega_{ci}}{v_\perp}\int\limits_{t_0}^{t}\frac{\partial\varphi}{\partial\bar{v}_\perp}dt_1 = \frac{e}{mv_\perp}\int\limits_{t_0}^{t}dt_1\int d\mathbf{k}\varphi(\mathbf{k},t_1)k_\perp(t_1)J_1\left(\frac{k_\perp(t_1)v_\perp}{\omega_{ci}}\right)e^{i\Psi}, \tag{46}$$

and $\delta v_\perp = 0$. In Eqs.(44)–(46), $\Psi = k_x X + k_y Y + k_z z + \mathbf{k}(t_1)\delta\mathbf{r}(t_1)$, and $i\mathbf{k}(t)\delta\mathbf{r}(t)$ denotes the phase shift resulted from perturbations of the ions orbits due to random ion-waves interactions,

$$\mathbf{k}(t)\delta\mathbf{r}(t) = k_x\delta X(t) + k_y\delta Y(t) - \frac{k_\perp(t)\bar{v}_\perp}{\omega_{ci}}\cos(\phi-\theta)\delta\phi(t). \tag{47}$$

In Eq.(47), the scattering of ions along the magnetic field is ignored. In variables $\bar{X}$, $\bar{Y}$, $\bar{\phi}$, $v_\perp$, the averaged over the time $t \gg \omega_{ci}^{-1}$ and over initial phases of the drift perturbations solution for $f_\alpha(t, k_x, k_y, k_z, v_\perp, \phi, v_z, z_1)$ in drift frequency range has a form

$$\begin{aligned} f_\alpha(t, k_x, k_y, k_z, v_\perp, \phi, v_z, z_1) &= i\frac{e_\alpha}{m_\alpha}\int^t dt_1\varphi(t_1, k_x, k_y, k_z) \\ &\times\exp\Big(-ik_z v_z(t-t_1) - \frac{1}{2}\left\langle(\mathbf{k}(t)\delta\mathbf{r}(t) - \mathbf{k}(t_1)\delta\mathbf{r}(t_1))^2\right\rangle\Big) \\ &\times J_0\left(\frac{k_\perp(t)v_\perp}{\omega_c}\right)J_0\left(\frac{k_\perp(t_1)v_\perp}{\omega_c}\right)\left[\frac{k_y}{\omega_{c\alpha}}\frac{\partial F_\alpha}{\partial X_\alpha} + k_{1z}\frac{\partial F_\alpha}{\partial v_z}\right] \\ &+f_\alpha(t=t_0, k_x, k_y, k_z, v_\perp\phi, v_z), \end{aligned} \tag{48}$$

in which $f_\alpha(t=t_0, \mathbf{k}, v_\perp, v_z)$ is the initial, determined at $t=t_0$ perturbation, assumed as independent on ϕ, of the distribution function F_α. In Eq.(48) Gaussian distribution for ions orbit disturbances is assumed. We use Eq.(48) in Poisson equation for the potential $\varphi(\mathbf{r}_\alpha, t)$,

$$\triangle\,\varphi(\mathbf{r}, t) = -4\pi\sum_{\alpha=i,e} e_\alpha\int f_\alpha(\mathbf{v}, \mathbf{r}, t)\,d\mathbf{v}_\alpha, \tag{49}$$

and obtain integral equation for separate spatial Fourier harmonic $\varphi(\mathbf{k}, t)$ for electrostatic potential, in which effect of the turbulent scattering of ions on sheared drift modes of random phases is accounted for,

$$\begin{aligned} k^2(t)\varphi(\mathbf{k}, t) &= \sum_{\alpha=i,e}\frac{i}{\lambda^2_{D\alpha}v^2_{T\alpha}}\int_{t_0}^t dt_1\varphi(\mathbf{k}, t_1)\int_0^\infty dv_\perp v_\perp\exp\left(-\frac{v_\perp^2}{v^2_{T\alpha}}\right) \\ \times J_0\left(\frac{k_\perp(t)v_\perp}{\omega_c}\right)&J_0\left(\frac{k_\perp(t_1)v_\perp}{\omega_c}\right)\exp\left(-\frac{1}{2}k_z^2v^2_{T\alpha}(t-t_1)^2 - \frac{1}{2}\left\langle(\mathbf{k}(t)\delta\mathbf{r}(t) - \mathbf{k}(t_1)\delta\mathbf{r}(t_1))^2\right\rangle\right) \\ &\times\left[k_y v_{d\alpha} + ik_z^2v^2_{T\alpha}(t-t_1)\right] - 4\pi\sum_{\alpha=i,e}e_\alpha\delta n_\alpha(\mathbf{k}, t, t_0). \end{aligned} \tag{50}$$

It follows from Eqs.(44)–(46), that for $k_x \sim k_y$ we have the estimates $\delta X \sim \delta Y$ and

$$\left|\frac{k_x\delta X(t)}{k_\perp(t)\dfrac{v_\perp}{\omega_c}\delta\phi(t)}\right| \sim \left|\frac{k_y^2 J_0\left(\dfrac{k_\perp(t_1)v_\perp}{\omega_{ci}}\right)}{k_\perp^2(t_1)J_1\left(\dfrac{k_\perp(t_1)v_\perp}{\omega_{ci}}\right)}\right|. \tag{51}$$

At initial times of the evolution, $t \ll (V_0')^{-1}$, for long wavelength perturbations with $k_\perp(t)v_\perp \ll \omega_{ci}$, the nonlinear phase shift is determined mainly by δX and δY. For short wavelength perturbations

with $k_\perp(t)\, v_\perp > \omega_{ci}$ for these times the $\delta\phi(t)$ term is important as well. At these times non-modal effects are negligible. At times $t > (V_0')^{-1}$ for $k_\perp(t)\, v_\perp \ll \omega_{ci}$

$$\left| \frac{k_x \delta X(t)}{k_\perp(t) \dfrac{v_\perp}{\omega_c} \delta\phi(t)} \right| \sim \frac{1}{k_y \rho_i} \frac{1}{(V_0' t)^3}, \tag{52}$$

and for $k_\perp(t)\, v_\perp > \omega_{ci}$,

$$\left| \frac{k_x \delta X(t)}{k_\perp(t) \dfrac{v_\perp}{\omega_c} \delta\phi(t)} \right| \sim \frac{1}{(V_0' t)^2}. \tag{53}$$

It follows from Eqs.(52), (53), that at times $t > (V_0')^{-1}$ turbulent scattering of the angle $\delta\phi(t)$ is the dominant process in the formation the turbulent shift of the phase of the electrostatic potential. Now consider the average phase shift term for times $(V_0' k_y \rho_i)^{-1} > t > (V_0')^{-1}$, but for $k_\perp v_\perp / \omega_{ci} < 1$. In that case $k_\perp(t) \approx k_y V_0' t$ and

$$J_1\left(\frac{k_\perp(t)\, v_\perp}{\omega_{ci}}\right) \approx \frac{k_\perp(t)\, v_\perp}{2\omega_{ci}} \approx \frac{k_y V_0' t v_\perp}{\omega_{ci}}. \tag{54}$$

In convective coordinates, solution for potential $\varphi(\mathbf{k}, t)$ at $k_y V_0' t \rho_i < 1$ (and $V_0' t > 1$) has a modal form

$$\varphi(\mathbf{k}, t) = \varphi(\mathbf{k}, t_0)\, e^{i\omega(\mathbf{k})t + \gamma(\mathbf{k})t}, \tag{55}$$

where $\omega(\mathbf{k})$ and $\gamma(\mathbf{k})$ are the frequency and growth rate of the kinetic drift instability. For times $t > (V_0')^{-1}$, the approximation

$$\begin{aligned}
\left\langle (\mathbf{k}(t)\, \delta\mathbf{r}(t) - \mathbf{k}(t_1)\, \delta\mathbf{r}(t_1))^2 \right\rangle &\approx \frac{v_\perp^2}{2\omega_{ci}^2} \left\langle (\mathbf{k}(t)\, \delta\phi(t) - \mathbf{k}(t_1)\, \delta\phi(t_1))^2 \right\rangle \\
&\approx \frac{v_\perp^2}{2\omega_{ci}^2} \left\langle (\mathbf{k}(t)\, (\delta\phi(t) - \delta\phi(t_1)))^2 \right\rangle \\
\approx \frac{e^2 k_y^2 (V_0')^6 v_\perp^2 t^2}{8\omega_{ci}^4 m_i^2} \int_{t_1}^{t} dt_1' \int_{t_1}^{t} dt_2' \int d\mathbf{k}_1 \left|\varphi(\mathbf{k}_1, t_0)\right|^2 k_{1y}^4 & \exp\left[\gamma(\mathbf{k}_1)(t_1' + t_2') + i\omega(\mathbf{k}_1)(t_1' - t_2')\right] \\
&\times (t_1' t_2')^2 \exp\left[-\frac{1}{2}\left\langle \mathbf{k}(t_1')\, (\delta\mathbf{r}(t_1') - \delta\mathbf{r}(t_2'))^2 \right\rangle\right]
\end{aligned} \tag{56}$$

is valid. With time variables $\tau = t_1 - t_2$ and $\widehat{t} = (t_1 + t_2)/2$, Eq.(56) becomes

$$\frac{v_\perp^2}{2\omega_{ci}^2} \left\langle (\mathbf{k}(t)\, (\delta\phi(t) - \delta\phi(t_1)))^2 \right\rangle = \frac{c^2 k_y^2 (V_0')^6 v_\perp^2 t^2}{8B^2 \omega_{ci}^2} \left(\int_{t_1 - t}^{0} d\tau \int_{t_1 - \frac{\tau}{2}}^{t + \frac{\tau}{2}} d\hat{t} + \int_{0}^{t - t_1} d\tau \int_{t_1 + \frac{\tau}{2}}^{t - \frac{\tau}{2}} d\hat{t} \right)$$

$$\times \int d\mathbf{k}_1 \left|\varphi\left(\mathbf{k}_1, t_0\right)\right|^2 k_{1y}^4 \hat{t}^4 e^{2\gamma(\mathbf{k}_1)\hat{t}+i\omega(\mathbf{k}_1)\tau} \exp\left[-\frac{c^2 k_{1y}^2 \left(V_0'\right)^6 v_\perp^2 t^2}{16 B^2 \omega_{ci}^2}\right.$$

$$\times \left(\int\limits_{-\tau}^{0} d\tau_1 \int\limits_{\hat{t}-\frac{1}{2}(\tau+\tau_1)}^{\hat{t}+\frac{1}{2}(\tau+\tau_1)} d\hat{t}_1 + \int\limits_{0}^{\tau} d\tau_1 \int\limits_{\hat{t}-\frac{1}{2}(\tau-\tau_1)}^{\hat{t}+\frac{1}{2}(\tau-\tau_1)} d\hat{t}_1\right) \int d\mathbf{k}_2 \left|\varphi\left(\mathbf{k}_2, t_0\right)\right|^2 k_{2y}^4 \hat{t}_1^4$$

$$\times \exp\left(2\gamma\left(\mathbf{k}_2\right)\hat{t}_1 + i\omega\left(\mathbf{k}_2\right)\tau_1\right.$$

$$\left.\left. -\frac{1}{2}\left\langle\left(\mathbf{k}\left(\hat{t}_1+\frac{1}{2}\tau_1\right)\left(\delta\mathbf{r}\left(\hat{t}_1+\frac{1}{2}\tau_1\right)-\delta\mathbf{r}\left(\hat{t}_1-\frac{1}{2}\tau_1\right)\right)\right)^2\right\rangle\right)\right]. \tag{57}$$

The integration over $\hat{t}_1$ is performed over narrow interval $\left(\hat{t}+\frac{1}{2}\left(\tau\pm\tau_1\right), \hat{t}-\frac{1}{2}\left(\tau\pm\tau_1\right)\right)$. In such a case the integrals over $\hat{t}_1$ may be approximately calculated as

$$\int\limits_{\hat{t}-\frac{1}{2}(\tau\pm\tau_1)}^{\hat{t}+\frac{1}{2}(\tau\pm\tau_1)} d\hat{t}_1 \hat{t}_1^4 e^{2\gamma(\mathbf{k}_2)\hat{t}_1 - f\left(\hat{t}_1,\tau\right)} \approx \left(\tau\pm\tau_1\right)\hat{t}^4 e^{2\gamma(\mathbf{k}_2)\hat{t} - f\left(\hat{t},\tau\right)}, \tag{58}$$

and for Eq.(57) we obtain

$$\frac{v_\perp^2}{2\omega_{ci}^2}\left\langle\left(\mathbf{k}\left(t\right)\left(\delta\phi\left(t\right)-\delta\phi\left(t_1\right)\right)\right)^2\right\rangle = \frac{c^2 k_y^2 \left(V_0'\right)^6 v_\perp^2 t^2}{8B^2\omega_{ci}^2}\left[\int\limits_{t_1-t}^{0} d\tau \int\limits_{t_1-\frac{\tau}{2}}^{t+\frac{\tau}{2}} d\hat{t} + \int\limits_{0}^{t-t_1} d\tau \int\limits_{t_1+\frac{\tau}{2}}^{t-\frac{\tau}{2}} d\hat{t}\right]$$

$$\int d\mathbf{k}_1 \left|\varphi\left(\mathbf{k}_1, t_0\right)\right|^2 k_{1y}^4 \hat{t}^4 e^{2\gamma(\mathbf{k}_1)\hat{t}+i\omega(\mathbf{k}_1)\tau}$$

$$\exp\left(-\tau\frac{c^2 k_{1y}^2 \left(V_0'\right)^6 v_\perp^2 \hat{t}^6}{8B^2\omega_{ci}^2}\int d\mathbf{k}_2 k_{2y}^4 \left|\varphi\left(\mathbf{k}_2, t_0\right)\right|^2 e^{2\gamma(\mathbf{k}_2)\hat{t}}\frac{C\left(\mathbf{k}_2, \hat{t}\right)}{\omega^2\left(\mathbf{k}_2\right)}\right), \tag{59}$$

where $C\left(\mathbf{k}_2, \hat{t}\right)$ resulted from infinite sequences of inner integration in exponential of the $\left\langle\mathbf{k}\left(\hat{t}_i+\frac{1}{2}\tau_i\right)\left(\delta\mathbf{r}\left(\hat{t}_i+\frac{1}{2}\tau_i\right)-\delta\mathbf{r}\left(\hat{t}_i-\frac{1}{2}\tau_i\right)\right)^2\right\rangle$ over $\hat{t}_i$ and determined by integral equation

$$C\left(\mathbf{k}_1, \hat{t}\right) = \frac{c^2 k_{1y}^2 \left(V_0'\hat{t}\right)^6 v_\perp^2}{8B^2\omega_{ci}^2}\int d\mathbf{k}_2 k_{2y}^4 \left|\varphi\left(\mathbf{k}_2, t_0\right)\right|^2 e^{2\gamma(\mathbf{k}_2)\hat{t}}\frac{C\left(\mathbf{k}_2, \hat{t}\right)}{\omega^2\left(\mathbf{k}_2\right)}. \tag{60}$$

Changing $\tau \to -\tau$ in first integral over τ in Eq.(59), we perform the integration over τ and obtain simple result

$$\frac{v_\perp^2}{2\omega_{ci}^2}\left\langle\left(\mathbf{k}\left(t\right)\left(\delta\phi\left(t\right)-\delta\phi\left(t_1\right)\right)\right)^2\right\rangle = 2t^2\int\limits_{t_1}^{t} d\hat{t}\frac{C\left(\mathbf{k}, \hat{t}\right)}{\hat{t}^2}. \tag{61}$$

V.NONLINEAR NON-MODAL EVOLUTION OF THE KINETIC DRIFT INSTABILITY IN SHEAR FLOW

Now we use Eq.(61) with $v_\perp^2$ changed on v_{Ti}^2 in Eq.(50) and integrate (50) over $v_\perp$. We obtain in ion terms additional multiplier, $\exp(-t^2\int\limits_{t_1}^{t} d\hat{t}\hat{t}^{-2}C\left(\mathbf{k}, \hat{t}\right))$, as compared with linear non-renormalised

equation(22),

$$k^2(t)\varphi(\mathbf{k},t)=\frac{i}{\lambda_{Di}^2}\int\limits_{t_0}^{t}dt_1\varphi(\mathbf{k},t_1)I_0\left(k_\perp(t)k_\perp(t_1)\rho_i^2\right)e^{-t^2\int\limits_{t_1}^{t}d\hat{t}\frac{C(\mathbf{k},\hat{t})}{\hat{t}^2}}e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)-\frac{1}{2}k_z^2v_{Ti}^2(t-t_1)^2}$$

$$\times\left[k_yv_{di}+ik_z^2v_{Ti}^2(t-t_1)\right]+\frac{i}{\lambda_{De}^2}\int\limits_{t_0}^{t}dt_1\varphi(\mathbf{k},t_1)I_0\left(k_\perp(t)k_\perp(t_1)\rho_e^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)-\frac{1}{2}k_z^2v_{Ti}^2(t-t_1)^2}$$

$$\times\left[k_yv_{de}+ik_z^2v_{Te}^2(t-t_1)\right]-4\pi\sum_{\alpha=i,e}e_\alpha\delta n_\alpha(\mathbf{k},t,t_0). \qquad (62)$$

For the perturbations of drift type, the equation for $\Phi(\mathbf{k},t)=\varphi(\mathbf{k},t)\Theta(\mathbf{k},t)$ has a form

$$\int\limits_{t_0}^{t}dt_1\left\{\frac{d}{dt_1}\Phi(\mathbf{k},t_1)(1+\tau)-\left[\frac{d}{dt_1}\left(\Phi(\mathbf{k},t_1)I_0\left(k_\perp(t)k_\perp(t_1)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}\right)\right.\right.$$

$$\left.\left.-\frac{d}{dt_1}\left(\Phi(\mathbf{k},t_1)\left(1-e^{-t^2\int\limits_{t_1}^{t}d\hat{t}\frac{C(\mathbf{k},\hat{t})}{\hat{t}^2}}\right)I_0\left(k_\perp(t)k_\perp(t_1)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}\right)\right]e^{-\frac{1}{2}k_z^2v_{Ti}^2(t-t_1)^2}\right\}$$

$$=i\int\limits_{t_0}^{t}dt_1\Phi(\mathbf{k},t_1)k_yv_{di}I_0\left(k_\perp(t)k_\perp(t_1)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}$$

$$-i\int\limits_{t_0}^{t}dt_1\Phi(\mathbf{k},t_1)k_yv_{di}I_0\left(k_\perp(t)k_\perp(t_1)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}\left(1-e^{-t^2\int\limits_{t_1}^{t}d\hat{t}\frac{C(\mathbf{k},\hat{t})}{\hat{t}^2}}\right)+$$

$$+\tau\int\limits_{t_0}^{t}dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1}+ik_yv_{de}\Phi(\mathbf{k},t_1)\right)e^{-\frac{1}{2}k_z^2v_{Te}^2(t-t_1)^2}. \qquad (63)$$

If $k_\perp\rho_i<1$ at time $t=0$ (at which the shear flow emerge), we will get $k_\perp(t)\rho_i<1$ on times $t<t_s$ throughout. By using the approximation

$$I_0\left(k_\perp(t)k_\perp(t_1)\rho_i^2\right)e^{-\frac{1}{2}\rho_i^2\left(k_\perp^2(t)+k_\perp^2(t_1)\right)}$$
$$\approx b_i+\left(k_xk_yV_0'(t+t_1)-\frac{1}{2}k_y^2(V_0')^2\left(t^2+t_1^2\right)\right)\rho_i^2\Theta(t), \qquad (64)$$

where $b_i=1-k_\perp^2\rho_i^2$, and $\Theta(t)$ indicates that the shear flow emerge at $t=0$, we present Eq.(63) in the form

$$\int\limits_{t_0}^{t}dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1}+i\omega(\mathbf{k})\Phi(\mathbf{k},t_1)\right)$$

$$=-\frac{b_i}{a_i}\int\limits_{t_0}^{t}dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1}+ik_yv_{di}\Phi(\mathbf{k},t_1)\right)\left(1-e^{-\frac{1}{2}k_z^2v_{Ti}^2(t-t_0)^2}\right)$$

$$-\frac{b_i}{a_i}\int\limits_{t_0}^{t}dt_1\left(\frac{d\Phi(\mathbf{k},t_1)}{dt_1}+ik_yv_{di}\Phi(\mathbf{k},t_1)\right)\left(1-\exp\left[-t^2\int\limits_{t_1}^{t}d\hat{t}\,\frac{C\left(\mathbf{k},\hat{t}\right)}{\hat{t}^2}\right]\right)$$

$$+\int_0^t dt_1\left(\frac{d\Phi\left(\mathbf{k},t_1\right)}{dt_1}+ik_yv_{di}\Phi\left(\mathbf{k},t_1\right)\right)\left(\frac{k_x}{k_y}\frac{\left(t+t_1\right)}{a_iV_0't_s^2}-\frac{\left(t^2+t_1^2\right)}{2a_it_s^2}\right)$$

$$+\int_0^t dt_1\Phi\left(\mathbf{k},t_1\right)\frac{1}{a_iV_0't_s^2}\left(\frac{k_x}{k_y}-V_0't\right)$$

$$+\frac{\tau}{a_i}\int_{t_0}^t dt_1\left(\frac{d\Phi\left(\mathbf{k},t_1\right)}{dt_1}+ik_yv_{de}\Phi\left(\mathbf{k},t_1\right)\right)\left(1-e^{-\frac{1}{2}k_z^2v_{Te}^2(t-t_0)^2}\right), \tag{65}$$

where $\omega\left(\mathbf{k}\right)$ is determined by Eq.(28). The right hand side of Eq.(65) is small for $\left(V_0'\right)^{-1}<t<t_s$, for $\tau<1$ and for weak ion Landau damping. Therefore, the solution to Eq.(65) we seek in the form

$$\Phi\left(\mathbf{k},t\right)=C\exp\left(-i\omega\left(\mathbf{k}\right)t+i\nu\left(\mathbf{k},t\right)\right). \tag{66}$$

Applying the procedure of the solution of the integral equation (25) to the renormalized version of that equation, (64), we obtain for $\left(V_0'\right)^{-1}<t<t_s$ the solution to Eq.(65) in the form

$$\Phi\left(\mathbf{k},t\right)=\Phi_0\exp\left[-i\omega\left(\mathbf{k}\right)t\left(1-\frac{1+\tau}{a_ib_i}\frac{t^2}{3t_s^2}\right)t+iRe\delta\omega\left(\mathbf{k}\right)t\right.$$
$$\left.+\left(\gamma\left(\mathbf{k}\right)-\frac{t}{2a_it_s^2}\right)t-\int_0^t C\left(\mathbf{k},t_1\right)dt_1\right], \tag{67}$$

where $C\left(\mathbf{k},t\right)$ is determined by the equation

$$C\left(\mathbf{k},t\right)=\frac{c^2}{B^2}k_y^2\rho_i^2\frac{\left(V_0't\right)^6}{8}\int d\mathbf{k}_1\left|\varphi\left(\mathbf{k}_1,t\right)\right|^2C\left(\mathbf{k}_1,t\right)\frac{k_{1y}^4}{\omega^2\left(\mathbf{k}_1\right)}. \tag{68}$$

If we omit linear non-modal terms in Eq.(67), the condition of the balance of the linear modal growth of the kinetic drift instability and non-linear non-modal dumping is determined by the equation $\gamma\left(\mathbf{k}\right)=C\left(\mathbf{k},t\right)$. By using this equation in Eq.(68), we obtain the equation, which determines the time, at which that balance occurs,

$$\frac{\gamma\left(\mathbf{k}\right)}{\left(V_0't\right)^6}=\frac{c^2}{8B^2}k_y^2\rho_i^2\int d\mathbf{k}_1\left|\varphi\left(\mathbf{k}_1,t\right)\right|^2\gamma\left(\mathbf{k}_1\right)\frac{k_{1y}^4}{\omega^2\left(\mathbf{k}_1\right)}. \tag{69}$$

The effect of the shear flow reveals in the reducing with time as $\left(V_0't\right)^{-6}$ the magnitude of the growth rate in the left part of the balance equation (69). That causes rapid suppression of the drift turbulence. The evolution of drift turbulence in times $t\geq t_s$ continues as strongly non-modal process, for which Markovian approximation, which is admissible for the solution Eq.(23) with small growth rate and non-modal terms with respect to the frequency $\omega\left(\mathbf{k}\right)$, ceases to be valid.

VI.CONCLUSIONS

In this paper, by using method of shearing modes or non-modal approach, originally developed by Lord Kelvin[15] for fluid descriptions of fluid shear flows, we develop for the first time non-modal

kinetic theory of plasma shear flow directed across the magnetic field. We obtain linear, (22), and renormalized nonlinear, (62), governing integral equations for the perturbed electrostatic potential. By using these equations we obtain linear, (32), (41), and renormalized nonlinear, (67), initial value problems solutions for kinetic drift instability of plasma shear flow. Obtained solutions display two distinct non-modal effects, which are observed in laboratory frame of reference.

The first effect displays the inhomogeneous Doppler shift, which is presented by the term $V_0' t k_y x$ in exponential of Eq.(33). This term displays the shearing of waves patterns in shear flow, observed in laboratory frame; in time $t > (k_y V_0'/k_x)^{-1}$, turbulence becomes almost one-dimensional in plane across the magnetic field and directed almost along the shear flow. It is obvious that this Doppler shift is irrelevant to the suppression of drift turbulence by shear flow.

Second non-modal effect is of principal importance for turbulence evolution in plasma shear flows. It reveals as a time dependent finite Larmor radius term in governed equations (22) and (62). The time dependence originates from the dot product of the time dependent coordinates of ion gyration in sheared coordinates, (Eqs.(12)), and wave number, which is time independent in these coordinates. That term completely conserves its form after the transformation to the laboratory frame variables, in which ion Larmor orbit is almost circular in shear flow with $|V_0'| \ll \omega_{ci}$ [19]. In laboratory frame of references, this non-modal effect is seemed as resulted from the coupling of the ion gyration and temporal variation of the wave number, $k_{\perp(lab)} = \left(k_y^2 + (k_x - k_y V_0' t)^2\right)^{1/2}$ of the shearing mode. Linear theory reveals (see Eqs.(32),(41)) that on the times $(V_0')^{-1} < t < (V_0' k_y \rho_i)^{-1}$ this time dependence resulted in the reducing the frequency and growth rate of the drift kinetic instability and to gradual suppression of the instability. It is important to note, that this effect is absent in the theory grounded on drift kinetic equation, in which effects of the finite Larmor radius are ignored. It is interesting to note that similar effect of the non–modal evolution of the perturbed electrostatic potential, which consists in reducing with time the frequency and growth rate, was discovered[14] in the investigations of the resistive drift instability on the base of the Hasegawa-Wakatani system. That effect for resistive drift instability originates from the time dependent polarization drift which is in fact the manifestation of the time dependent effect of the finite ion Larmor radius.

Decisive impact on the temporal evolution of the kinetic drift instability has non-linear non–modal effect of the turbulent scattering of ions by the ensemble of sheared waves. We find that turbulent scattering of the gyrophase of ion Larmor orbit is the dominant effect, which determines extremely rapid suppression of drift turbulence by flow shear.

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Appendix 1.Equilibrium distribution function $F_{0\alpha}$.

The equilibrium distribution function $F_{0\alpha}(x, v_x, v_y, v_z)$ is governed by the equation

$$v_x \frac{\partial F_{0\alpha}}{\partial x} + \left(\frac{e_\alpha}{m_\alpha} E_0(x) + \omega_{c\alpha} v_y\right) \frac{\partial F_{0\alpha}}{\partial v_x} - \omega_{c\alpha} v_x \frac{\partial F_{0\alpha}}{\partial v_y} = 0. \tag{A.1}$$

From characteristic equations

$$\frac{dx}{v_x} = \frac{dv_x}{\frac{e}{m} E_0(x) + \omega_{c\alpha} v_y} = -\frac{dv_y}{\omega_{c\alpha} v_x} \tag{A.2}$$

the relation

$$-\frac{e}{m} \omega_c E_0(x)\, dx + \omega_{c\alpha} v_y dv_y + \omega_{c\alpha} v_x dv_x = d(\omega_c H) = 0 \tag{A.3}$$

follows in which H is Hamiltonian of a particle in electric and magnetic fields. By using the expansions

$$E_0(x) = E_0(X_\alpha) + E_0'(X_\alpha)(x - X_\alpha), \tag{A.4}$$

and

$$v_y = v_{y\alpha} + V_0(X_\alpha) + V_0'(X_\alpha)(x - X_\alpha), \tag{A.5}$$

and accounting for that $V_0(X_\alpha) = -cE_0(X_\alpha)/B_0$ and $dv_y = -\omega_{c\alpha} dx$, we obtain that

$$-\omega_{c\alpha}^2 v_{\alpha y} dx_\alpha + \omega_{c\alpha} v_{x\alpha} dv_{x\alpha} = d(\omega_c H) = 0, \tag{A.6}$$

where identities $x = x_\alpha$ and $v_x = v_{x\alpha}$ were used. It follows from $(A.5)$ that $dv_y = dv_{y\alpha} + V_0'(X_\alpha)\, dx_\alpha$. Therefore $dx_\alpha = dx = -dv_y/\omega_{c\alpha} = -(dv_{y\alpha} + V_0'(X_\alpha)\, dx_\alpha)/\omega_{c\alpha}$ and $dx_\alpha = -dv_{y\alpha}/\mu\omega_{c\alpha}$. With Eqs.(13) it follows that

$$\frac{1}{2} d\left(\omega_{c\alpha} v_\perp^2\right) = d(\omega_c H) = 0. \tag{A.7}$$

The same conclusion about absence of the spatial dependence in Hamiltonian in sheared coordinates follows from the analysis of the characteristics of Eq.(8).

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